Quantum-classical correspondence for resonances on vector bundles

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Outline

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$\mathcal{M}$ compact connected Riemannian manifold, no boundary

configuration space of a point particle
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configuration space of a point particle

$T^*\mathcal{M} \xrightarrow{\pi} \mathcal{M}$ cotangent bundle, phase space

$T^*\mathcal{M} \supset S^*\mathcal{M}$ cosphere bundle, momentum $= 1$
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$\varphi_t : S^*\mathcal{M} \to S^*\mathcal{M}$ geodesic flow, free motion

$X : C^\infty(S^*\mathcal{M}) \to C^\infty(S^*\mathcal{M})$ generating vector field of $\varphi_t$
\( \mathcal{M} \) compact connected Riemannian manifold, no boundary

*configuration space* of a point particle

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\[ \Delta_\mathcal{M} : C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \text{ Laplace-Beltrami op. (positive)} \]
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\( \Delta_{\mathcal{M}} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \) Laplace-Beltrami op. (positive)

correspondence principle for \( X \) and \( \Delta_{\mathcal{M}} \)
How to compare the "classical" operator

\[ X : C^\infty(S^*\mathcal{M}) \to C^\infty(S^*\mathcal{M}) \]

with the "quantum operator"

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How to compare the "classical" operator

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\[ \Delta_M : C^\infty(M) \to C^\infty(M) \]

Pushforward

\[ \pi_* : C^\infty(S^*M) \to C^\infty(M) \quad (\text{fiber integration}) \]
Motivation: “Trivial” example

\[ \mathcal{M} := S^1 = \mathbb{R}/(2\pi\mathbb{Z}) = \{ e^{i\phi} : \phi \in \mathbb{R} \} \]
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\[ \mathcal{M}_+ \quad \mathcal{M} \quad \mathcal{M}_- \]

\[ \pi \]

\[ \mathcal{M} \]

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\[ \begin{array}{ccc}
\mathcal{M}_+ & \circ & \mathcal{M}_- \\
\downarrow \pi & & \downarrow \\
\mathcal{M} & \circ & \mathcal{M}
\end{array} \]

\[ C^\infty(S^* \mathcal{M}) = C^\infty(\mathcal{M}_+) \oplus C^\infty(\mathcal{M}_-) \]

\[ X = \pm \frac{\partial}{\partial \phi} \text{ on } C^\infty(\mathcal{M}_\pm) \]

\[ \Delta \mathcal{M} = -\frac{\partial^2}{\partial \phi^2} \]
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\[ \mathcal{M}_+ \bigcirc \mathcal{M}_- \bigcirc \mathcal{M} \]

\[ \pi \]

\[ \{(e^{ik\phi}, e^{il\phi})\}_{k,l \in \mathbb{Z}} \text{ orthogonal basis of } L^2(S^*\mathcal{M}) \]
Motivation: “Trivial” example

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\pi & & & \pi \\
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\[ \mathcal{M}_+ \quad \bigcirc \quad \bigcirc \quad \mathcal{M}_- \quad \text{C}^\infty(S^*\mathcal{M}) = C^\infty(\mathcal{M}_+) \oplus C^\infty(\mathcal{M}_-) \]

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\[ \pi_*(e^{i k \phi}, e^{-i k \phi}) = e^{i k \phi} + e^{-i k \phi} = 2 \cos(k \phi) \in \text{Eig}(\Delta_\mathcal{M}, k^2) \]

\[ \pi_*(e^{i k \phi}, -e^{-i k \phi}) = e^{i k \phi} - e^{-i k \phi} = 2i \sin(k \phi) \in \text{Eig}(\Delta_\mathcal{M}, k^2) \]
Spectra of $\Delta_M$ and $X$ for $M = S^1$

Pushfwd $\pi_* : C^\infty(S^*M) \to C^\infty(M)$ induces isomorphisms

$$\text{Eig}(X, ik) \overset{\pi_*}{\cong} \text{Eig}(\Delta_M, k^2), \quad k \in \mathbb{Z}$$

$$\text{Spectrum}(X) = \{ \bullet \} = i\mathbb{Z}$$

$$\text{Spectrum}(\Delta_M) = \{ \blacksquare \} = \{ k^2 \}_{k \in \mathbb{N}_0}$$
General situation: The Anosov condition

\[ \dim \mathcal{M} > 1 \implies X \text{ not elliptic. Require additional condition} \]
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The geodesic flow \( \varphi_t : S^*\mathcal{M} \to S^*\mathcal{M} \) is Anosov if there is a flow-invariant decomposition

\[
T(S^*\mathcal{M}) = E_0 \oplus E_+ \oplus E_- , \quad E_0 = \text{span}(X),
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s.t. \( E_\pm \) are continuous and there are \( \lambda, C > 0 \) with

\[ \| D\varphi_\pm t v \| \leq C e^{-\lambda t} \| v \| \quad \forall \, v \in E_\pm , \, t \geq 0 . \]
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\( E_+ : \text{exponentially stable bundle, } E_- : \text{exp. unstable bundle} \)
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\( E_\pm \) in general not smooth
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An Anosov flow is \textit{chaotic} with positive and negative Lyapunov exponents in \( E_+ \) and \( E_- \), respectively
The Anosov condition

Theorem (Anosov, Anosov-Sinai 1967)

If $\mathcal{M}$ has strictly negative sectional curvatures, the geodesic flow $\varphi_t$ on $S^*\mathcal{M}$ is Anosov.
The Anosov condition

Theorem (Anosov, Anosov-Sinai 1967)

*If $\mathcal{M}$ has strictly negative sectional curvatures, the geodesic flow $\varphi_t$ on $S^*\mathcal{M}$ is Anosov.*

Example: Compact hyperbolic manifolds

$$\mathcal{M} = \mathcal{H}^{n+1}/\Gamma$$
The Anosov condition

Theorem (Anosov, Anosov-Sinai 1967)

If \( \mathcal{M} \) has strictly negative sectional curvatures, the geodesic flow \( \varphi_t \) on \( S^*\mathcal{M} \) is Anosov.

Example: Compact hyperbolic manifolds

\[
\mathcal{M} = \mathcal{H}^{n+1} / \Gamma = \Gamma \backslash \text{SO}(n + 1, 1)_0 / \text{SO}(n + 1),
\]

\( \Gamma \subset \text{SO}(n + 1, 1)_0 \) discrete, torsion-free, cocompact
Classical resonances and resonant states

From now on, assume $\varphi_t$ Anosov
Classical resonances and resonant states

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**Theorem (Liverani 2004)**

*There are Hilbert spaces $\subset D'(S\ast M)$ in which $X$ has discr. spectrum consisting of eigenvalues of finite multiplicities*
Classical resonances and resonant states

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**Theorem (Liverani 2004)**

There are Hilbert spaces $\subset \mathcal{D}'(S^*M)$ in which $X$ has discr. spectrum consisting of eigenvalues of finite multiplicities

Eigenvalues: *classical (Pollicott-Ruelle) resonances*

Eigenvectors $\in \mathcal{D}'(S^*M)$: *resonant states*
Classical resonances and resonant states

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Eigenvalues: classical (Pollicott-Ruelle) resonances
Eigenvalues $\in D'(S^*M)$: resonant states

Spectral invariant of chaotic dynamical system $(S^*M, \varphi_t)$
Resonance distribution for pinched curvature

Theorem (Faure-Tsujii 2013)

For pinched sectional curvature

\[-\frac{1}{C} > \kappa > -C \sim -1:\]

Image source: T. Weich
Classical resonances on hyperbolic manifolds

Theorem (Dyatlov, Faure, Guillarmou 2013)

\[ \kappa = -1, \text{ i.e., } M \text{ hyperbolic:} \]
Classical resonances on hyperbolic manifolds

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Proof involves resonant states on vector bundles
Classical resonances on vector bundles

\[ \nabla^V, \nabla^W \]

complex v.b. with connections \( \nabla^V, \nabla^W \)
Classical resonances on vector bundles

\[ V \xrightarrow{\nabla_V} W \]

complex v.b. with connections \( \nabla_V, \nabla_W \)

\( S^* M \xrightarrow{\nabla^*} M \)

\[ \Delta_W := \Delta_{\nabla_W, \nabla^L.C.} : \Gamma^\infty(W) \to \Gamma^\infty(W) \] Bochner Laplacian

\[ X_V := \nabla^V_X : \Gamma^\infty(V) \to \Gamma^\infty(V) \] covariant derivative
Classical resonances on vector bundles

\[ \mathcal{V} \overset{\mathcal{W}}{\longrightarrow} \text{complex v.b. with connections } \nabla^\mathcal{V}, \nabla^\mathcal{W} \]

\[ S^*M \overset{M}{\longrightarrow} \]

\[ \Delta_\mathcal{W} := \Delta_{\nabla^\mathcal{W}, \nabla^{\text{L.C.}}} : \Gamma^\infty(\mathcal{W}) \to \Gamma^\infty(\mathcal{W}) \quad \text{Bochner Laplacian} \]

\[ X_\mathcal{V} := \nabla^\mathcal{V}_X : \Gamma^\infty(\mathcal{V}) \to \Gamma^\infty(\mathcal{V}) \quad \text{covariant derivative} \]

**Lemma (Definition)**

For \( \lambda \in \mathbb{C} \), the set of resonant states on \( \mathcal{V} \) is

\[ \text{Res}(X_\mathcal{V}, \lambda) = \{ s \in D'(S^*M, \mathcal{V}) : X_\mathcal{V} s = \lambda s, \ WF(s) \subset E^*_+ \} . \]

If \( \text{Res}(X_\mathcal{V}, \lambda) \neq \{0\} \), \( \lambda \) is called classical resonance on \( \mathcal{V} \).
Examples of interesting vector bundles

\[ \mathcal{M} = \mathcal{H}^{n+1} / \Gamma = \Gamma \backslash \text{SO}(n + 1, 1)_0 / \text{SO}(n + 1), \]
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\[ \mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma \backslash \text{SO}(n+1,1)/\text{SO}(n+1), \]

\[ \mathcal{V} = \Lambda^k T^*(S^*\mathcal{M}) \quad \text{or} \quad \mathcal{V} = \otimes_{s,\text{tr}=0}^k T^*_{S^\perp}(S^*\mathcal{M}) \]

or \[ \mathcal{V} = \Lambda^k E^*_\pm \quad (\text{here } E^*_\pm \text{ are smooth}) \]
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More generally, \[ \mathcal{V} = \mathcal{V}_\tau \text{ ass. to unitary rep. } \tau \text{ of SO}(n) \]

using \[ S^*\mathcal{M} = \Gamma \backslash \text{SO}(n+1,1)_0/\text{SO}(n) \]
Examples of interesting vector bundles

\[ M = \mathcal{H}^{n+1}/\Gamma = \Gamma\backslash\text{SO}(n+1,1)_0/\text{SO}(n+1), \]

\[ \mathcal{V} = \bigwedge^k T^*(S^*M) \quad \text{or} \quad \mathcal{V} = \bigotimes_{s,\text{tr}=0}^k T^*_{S^\perp}(S^*M) \]

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More generally, \( \mathcal{V} = \mathcal{V}_\tau \) ass. to unitary rep. \( \tau \) of \( \text{SO}(n) \)
using \( S^*M = \Gamma\backslash\text{SO}(n+1,1)_0/\text{SO}(n) \)

More generally, for \( M = \Gamma\backslash G/K \) Riem. loc. symm. of rk. 1,
\( \mathcal{V} = \mathcal{V}_\tau \) for unitary rep. \( \tau \) of certain subgroup \( M \subset K \)
using \( S^*M = \Gamma\backslash G/M \)
For $\mathcal{M} = \Gamma \backslash G / K$ cpt. Riem. loc. symm. of rk. one, $\mathcal{V} = \mathcal{V}_\tau$:

**Theorem (T. Weich, B.K., arXiv:1710.04625 (v2 2018))**

The classical resonances on $\mathcal{V}$ outside of the real axis lie in exact lines.

*Image source: T. Weich*
Band structure result

For $\mathcal{M} = \Gamma \backslash G / K$ cpt. Riem. loc. symm. of rk. one, $\mathcal{V} = \mathcal{V}_\tau$:

**Theorem (T. Weich, B.K., arXiv:1710.04625 (v2 2018))**

*The classical resonances on $\mathcal{V}$ outside of the real axis lie in exact lines.*

*Trivial 1-dim. $\tau$ gives scalar result for all compact Riemannian locally symmetric spaces of rank one*

*Image source: T. Weich*
First band resonant states and pushforwards

Special role played by first band resonant states

\[ \text{Res}^0(X_\mathcal{V}, \lambda) \subset \text{Res}(X_\mathcal{V}, \lambda) \]
First band resonant states and pushforwards

Special role played by first band resonant states

\[ \text{Res}^0(X_V, \lambda) \subset \text{Res}(X_V, \lambda) \]

\[ \text{Res}^0(X_V, \lambda) = \left\{ s \in \mathcal{D}'(S^*M, V) : X_V s = \lambda s, \quad \nabla_Y s = 0 \ \forall \ Y \in \Gamma(E_-) \right\} \]
First band resonant states and pushforwards

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Want to find a bundle \( \mathcal{W} \) over \( M \) and a pushforward

\[ \pi_* : \mathcal{D}'(S^*M, V) \to \mathcal{D}'(M, \mathcal{W}) \] to ask
First band resonant states and pushforwards

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Q For \( s \in \text{Res}^0(X_V, \lambda) \): Is \( \pi_* s \) an eigensection of \( \Delta_{\mathcal{W}} \) for some eigenvalue \( \mu(\lambda) \in \mathbb{R} \)?
First band resonant states and pushforwards

Special role played by first band resonant states

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\[ \text{Res}^0(X_V, \lambda) = \left\{ s \in D'(S^*M, V) : X_V s = \lambda s, \quad \nabla_Y s = 0 \ \forall \ Y \in \Gamma(E_+) \right\} \]

Want to find a bundle \( \mathcal{W} \) over \( M \) and a pushforward

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Q Is there an Isom. \( \pi_* : \text{Res}^0(X_V, \lambda) \cong \text{Eig}(\Delta_{\mathcal{W}}, \mu(\lambda)) \) ?
Examples of compatible bundles

When $\mathcal{V}$ is a subbundle of $\bigotimes^k T^*(S^*\mathcal{M})$, for example

$$\mathcal{V}_1 = \Lambda^k T^*(S^*\mathcal{M}), \quad \mathcal{V}_2 = \Lambda^k E^*, \quad \mathcal{V}_3 = \bigotimes^k_{s, \text{tr}=0} T^*_{S^\perp} (S^*\mathcal{M})$$
Examples of compatible bundles

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\]

put \( \mathcal{W} := (\bigotimes^k D\pi^t)^{-1}(\mathcal{V}) \subset \bigotimes^k T^*M \quad (\pi : S^*M \to M) \)
Examples of compatible bundles

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E.g. $\mathcal{W}_1 = \mathcal{W}_2 = \Lambda^k T^*M, \quad \mathcal{W}_3 = \bigotimes^k_{s, \text{tr}=0} T^*M$
Examples of compatible bundles

When $\mathcal{V}$ is a subbundle of $\otimes^{k} T^{*}(S^{*} \mathcal{M})$, for example

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\mathcal{V}_1 = \Lambda^{k} T^{*}(S^{*} \mathcal{M}), \quad \mathcal{V}_2 = \Lambda^{k} E_{\pm}, \quad \mathcal{V}_3 = \otimes_{s, \text{tr}=0}^{k} T_{S_{\perp}}^{*}(S^{*} \mathcal{M})
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put $\mathcal{W} := (\otimes^{k} D\pi^{t})^{-1}(\mathcal{V}) \subset \otimes^{k} T^{*} \mathcal{M}$ \quad ($\pi : S^{*} \mathcal{M} \to \mathcal{M}$)

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Then there is a natural pushforward

$$
\pi_{\ast} : \mathcal{D}'(S^{*} \mathcal{M}, \mathcal{V}) \to \mathcal{D}'(\mathcal{M}, \mathcal{W})
$$
Tensor bundles on hyperbolic manifolds

For $\mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma \backslash \text{SO}(n+1,1)_0/\text{SO}(n+1)$:

Theorem (Dyatlov, Faure, Guillarmou 2013)

For all $\lambda \in \mathbb{C}$ outside a discrete set $\mathcal{A} \subset \mathbb{R}$, there is an iso.

$$\pi_\ast : \text{Res}^0(X_{s,\text{tr}=0} \otimes^k T^*_{S\perp}(S^*\mathcal{M}), \lambda) \xrightarrow{\cong} \text{Eig}((\Delta_{s,\text{tr}=0} \otimes^k \mathcal{M}, \mu(\lambda)) \cap \ker \text{div}$$
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\pi_* : \text{Res}^0(X_{\otimes^k s, \text{tr}=0} T^*_{S \perp} (S^* \mathcal{M}), \lambda) \xrightarrow{\cong} \text{Eig}(\Delta_{\otimes^k s, \text{tr}=0} T^* \mathcal{M}, \mu(\lambda)) \cap \ker \text{div}
$$

$\mu(\lambda)$ is an explicitly given quadratic polynomial in $\lambda$
Tensor bundles on hyperbolic manifolds
For $\mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma \backslash \text{SO}(n+1,1)_0/\text{SO}(n+1)$:

**Theorem (Dyatlov, Faure, Guillarmou 2013)**

*For all $\lambda \in \mathbb{C}$ outside a discrete set $\mathcal{A} \subset \mathbb{R}$, there is an iso.*

$$
\pi_* : \text{Res}^0(\chi_{s,\text{tr}=0}^k T^*_S (S^* \mathcal{M}), \lambda) \\
\cong \text{Eig}(\Delta_{s,\text{tr}=0}^k T^* \mathcal{M}, \mu(\lambda)) \cap \ker \text{div}
$$

$\mu(\lambda)$ is an explicitly given quadratic polynomial in $\lambda$

Moreover,

$$
\text{Res}^0(\chi_{s,\text{tr}=0}^k T^*_S (S^* \mathcal{M}), \lambda) \cong \text{Res}^m(\chi, \lambda - m)
$$

$m$-th band
Result for general associated bundles

For $\mathcal{M} = \Gamma \backslash G/K$ cpt. Riem. loc. symm. of rk. one, $\mathcal{V} = \mathcal{V}_\tau$, $\mathcal{W} = \mathcal{W}_\sigma$ for appropriate irreducible rep. $\tau$ and $\sigma$:
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Theorem (T. Weich, B.K., preprint 2018)

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*for some differential operators $D_1, \ldots, D_N$.***
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Classical resonances and topology

For $\mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma\backslash \text{SO}(n + 1, 1)_0/\text{SO}(n + 1)$:
Classical resonances and topology

For \( \mathcal{M} = \mathcal{H}^{n+1}/\Gamma = \Gamma \backslash \text{SO}(n+1,1)_0/\text{SO}(n+1) \):

**Theorem (T. Weich, B.K., preprint 2018)**

One has

\[
\dim_{\mathbb{C}} \text{Res}^0(\Lambda^p \mathcal{E}^+, 0) = \begin{cases} 
2 b_p(\mathcal{M}), & p \neq \frac{n}{2}, \\
 b_p(\mathcal{M}), & p = \frac{n}{2},
\end{cases}
\]

where \( b_p(\mathcal{M}) = \dim_\mathbb{C} H^p(\mathcal{M}, \mathbb{C}) \) is the \( p \)-th Betti number.

Similar result proved by Dyatlov and Zworski in dimension 2 and *variable negative curvature*
Usual Sobolev spaces: \( s \in \mathbb{R} \),

\[
m_s : T^* M \to \mathbb{R}, \quad \xi \mapsto (1 + \|\xi\|^2)^{-s/2}
\]
growth-/symbol function.
Usual Sobolev spaces: $s \in \mathbb{R}$,

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growth-/symbol function. Choose quantization map

$$\text{Op} : \{\text{symbol functions } T^* M \to \mathbb{R}\} \quad \longrightarrow \quad \{\text{operators } D(M) \to D'(M)\}$$
Usual Sobolev spaces: \( s \in \mathbb{R} \),

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\[ \text{Op} : \{ \text{symbol functions } T^* \mathcal{M} \to \mathbb{R} \} \]
\[ \to \{ \text{operators } \mathcal{D}(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}) \} \]

For \( s > 0 \): Obtain \( \text{Op}(m_s) : L^2(\mathcal{M}) \to L^2(\mathcal{M}) \),

\[ H^s(\mathcal{M}) := \text{Op}(m_s)(L^2(\mathcal{M})) \]
Usual Sobolev spaces: $s \in \mathbb{R}$,

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Then

$$\Delta_\mathcal{M} - \lambda : H^2(\mathcal{M}) \to L^2(\mathcal{M})$$

is a Fredholm operator for $\lambda \in \mathbb{C}$
Usual Sobolev spaces: $s \in \mathbb{R}$,

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$$H^s(M) := \text{Op}(m_s)(L^2(M))$$

Then

$$\Delta_M - \lambda : H^2(M) \to L^2(M)$$

is a Fredholm operator for $\lambda \in \mathbb{C}$

Analytic Fredholm theory $\implies$ spectrum($\Delta$) discrete in $\mathbb{C}$
Anisotropic Sobolev spaces

Faure-Sjöstrand 2011: \( \exists \ m \in \mathcal{C}^\infty(T^*(S^*\mathcal{M}), [-1, 1]): \)

\[ X \ m \leq 0, \quad m \equiv \pm 1 \text{ near } E^*_\pm \subset T^*(S^*\mathcal{M}) \]
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Define

\[ \tilde{m}_s(\xi) := (1 + \|\xi\|^2)^{-sm(\xi)/2}, \quad \xi \in T^*(S^*\mathcal{M}), \ s \in \mathbb{R} \]
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\[
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\]

Define

\[
\tilde{m}_s(\xi) := (1 + \|\xi\|^2)^{-sm(\xi)/2}, \quad \xi \in T^*(S^*M), s \in \mathbb{R}
\]

For \( s > 0 \), obtain \( \text{Op}(\tilde{m}_s) : L^2(S^*M) \to L^2(S^*M) \),

\[
H^s_{\text{an}} := \text{Op}(\tilde{m}_s)(L^2(S^*M))
\]
Resonances as poles of the resolvent

Theorem (Liverani 2005 / Faure-Sjöstrand 2011)

For $C_0 > 0$ we find $s > 0$ such that $X - \lambda : D^{s}_{an} \rightarrow H^{s}_{an}$ is for $\Re \lambda > -C_0$ a Fredholm operator.
Resonances as poles of the resolvent

Theorem (Liverani 2005 / Faure-Sjöstrand 2011)

For $C_0 > 0$ we find $s > 0$ such that $X - \lambda : D^s_{\text{an}} \to H^s_{\text{an}}$ is for $\Re \lambda > -C_0$ a Fredholm operator and $\exists C_1 > 0$ such that

$$(X - \lambda)^{-1} : H^s_{\text{an}} \to H^s_{\text{an}}$$

exists for $\Re \lambda > C_1$. 
Resonances as poles of the resolvent

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$$\mathbb{C} \supset \{\Re \lambda > C_1\} \ni \lambda \rightarrow (X - \lambda)^{-1} : H^s_{\text{an}} \rightarrow H^s_{\text{an}}$$

has a meromorphic continuation to $\{-C_0 < \Re \lambda\} \subset \mathbb{C}$ with poles of finite ranks.
Resonances as poles of the resolvent

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Resonances as poles of the resolvent

Theorem (Liverani 2005 / Faure-Sjöstrand 2011)

For $C_0 > 0$ we find $s > 0$ such that $X - \lambda : D^s_{an} \to H^s_{an}$ is for
$\text{Re} \lambda > -C_0$ a Fredholm operator and $\exists C_1 > 0$ such that

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poles of finite ranks. Poles (with rank) and residues are
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Poles are classical resonances, residues are eigenspaces with
resonant states $\in H^s_{an} \subset \mathcal{D}'(S^*\mathcal{M})$.