The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds

$\varphi_t : M \to M$ smooth Anosov flow, $M$ cpt. mfd., $TM = E_o \oplus E_s \oplus E_u$, $E_{s/u}$ orient.

$\zeta_R(z) := \prod_{\gamma \text{ prim.}} (1 - e^{-2\pi T_\gamma})$, $\Im z \gg 1$ Ruelle zeta function (untwisted)

$\zeta$ : primitive closed orbits, $T_\gamma$ : period of $\gamma$

$\zeta_R$ extends meromorphically to $\mathbb{C}$ (Giuliani-Liverani-Pollicott 2013 / Dyatlov-Zworski 2016)

Interesting quantity: order of vanishing of $\zeta_R$ at $0 \in \mathbb{C}$

$\exists ! \; n \in \mathbb{Z} : z^n \zeta_R(z)$ holomorphic and non-zero near $0$

$n = n(\varphi_t)$ numerical invariant of $\varphi_t$ determined by period spectrum $\{T_\gamma\}$

Choice of $0 \in \mathbb{C}$ justified by distinguished invariance properties of $\zeta_R$ at $0$
The invariant $n(\Psi_t)$ can be studied for $\Psi_t$ in various classes of Anosov flows:

\[
\{\text{Anosov flows}\} \supset \{\text{volume-preserving flows}\} \supset \{\text{contact flows}\} \supset \{\text{geodesic flows}\}
\]

\[
(M, \Psi_t), \quad \Psi_t^* \mathcal{R} = \mathcal{R}.
\]

\[
n(\Psi_t) = n(\Psi_t, \mathcal{R})
\]

\[
(M, \alpha), \quad L_x \alpha = 0,
\]

\[
\Psi_t : \text{flow of } \alpha
\]

\[
(\mathcal{R}_x, g), \quad M = S^1 \Sigma
\]

\[
\Psi_t : \text{geodesic flow}
\]

\[
n(\Psi_t) = n(g)
\]

1) Results in dimension 3

**Thm. (Dyatlov-Zworski 2017):** (M, \alpha) 3-dim. cpt. conn. contact mfld., $\Psi_t : M \to M$ contact Anosov with $E_5, E_6$ orientable

$\Rightarrow$ $n(\Psi_t) = n(\alpha) = b_1(M) - 2$

first Betti number of $M$

In particular, $n(\Psi_t) = n(\alpha)$ independent of $\alpha$ and $M$ as long as $b_1$ is fixed.
Corollary: \( M = S \Sigma, (\Sigma, g) \) cpt. conn. orient. Riem. surface of curvature \( \kappa \leq 0 \),

\( \Psi_t \) geodesic flow

\[ \Rightarrow \quad n(\Psi_t) = n(g) = b_\tau(M) - 2 = -\kappa(\Sigma) \]

Every characteristic

In particular, \( n(\Psi_t) = n(g) \) independent of \( g \) - and of \( \Sigma \) - for fixed \( \tau \)

Conversely, \( \tau \) \& genus determined by geodesic length spectrum

Thm. (Cekić–Paternain 2019): (\( M, \alpha \)) cpt. conn. 3-dim. contact mfld., \( \chi \in C^\infty(M; TM) \)

Reeb vector field generating an Anosov flow \( \Psi_t \) with
\( E_u, E_s \) orientable, \( \Sigma := \alpha \wedge d\alpha \) volume form

\( \Rightarrow \exists \chi \in C^0(M; TM), \ \varepsilon > 0: \ \left| \chi - \frac{\varepsilon}{2} \right| \Sigma = 0 \)

Flow \( \Psi_t^\varepsilon \) generated by \( X_t := \chi + \varepsilon \gamma, \ \varepsilon \in \mathbb{R} \)

Satisfies
\[ n(\Psi_t^\varepsilon) = \begin{cases} b_\tau(M) - 2, & \varepsilon = 0 \\ b_\tau(M) - 3, & \varepsilon \neq 0, \ |\varepsilon| < 3 \end{cases} \]

\( n(\Psi_t) = n(\Psi_t^\varepsilon) \) can jump under small perturbations of \( \Psi_t \), even for \( \Sigma \) fixed
2) Higher dimensions (mostly dimension 5)

**Thm. (Fried 1986):** \( M = S \Sigma, (\Sigma, g_\mathfrak{h}) \) hyperbolic \((\tau = -1)\) oriented mfld. of dim. \( 2k+1 \), \( \gamma_+ \) geodesic flow

\[ \Rightarrow n(\gamma_+) = n(g_\mathfrak{h}) = (2k+2)\beta_0(\Sigma) - 2k\beta_1(\Sigma) + (2k-2)\beta_2(\Sigma) - \ldots + (-1)^k 2\beta_k(\Sigma) \]

\( \tau \) suggests that \( n(\gamma_+) = n(g) \) might be a topological invariant for general \( g \) with \( \tau < 0 \), given by the above combination of Betti numbers.

Generalizations to variable curvature so far concentrated on Fried's conjecture for the twisted Ruelle zeta function \( \mathfrak{Z}_{\mathfrak{R}, \phi} \) associated with an acyclic (unitary) representation \( \phi: \pi_1(M) \to U(\mathfrak{g}) \), i.e., \( b_k(M, E\phi) = \dim H^k(M; E\phi) = 0 \) \( \forall k \geq 0 \).

\( \text{de Rham cohomology with values in flat bundle } E\phi \text{ def. by } \phi \)
Theorem (Dang-Guillarmou-Rivièr-Shen 2018): \( M = S \Sigma, (\Sigma, g_{\Sigma}) \) cpt. orient. hyperbolic 3-mfld., 
\( \mathcal{P}: \Pi_1(M) \to U(\mathbb{C}^r) \) acyclic unitary rep., 
\( X_0 \in \mathcal{E}^\infty(M, TM) \) geodesic vector field

\[ \exists \mathcal{U} \subset \mathcal{E}^\infty(M, TM) \text{ open, } \forall x \in \mathcal{U}, : \]

\( \forall x \in \mathcal{U}, \) the flow \( \mathcal{P} \) generated by \( x \) 
is Anosov and satisfies

\[ R_{\mathcal{P}}(0) = \mathcal{T}_{\text{Reid}}(M) \]

\( R \) Reidemeister torsion of \( M \), topological invariant

Our case

\( \mathcal{R} \) corresponds to \( \mathcal{R}, r_0 \) with \( \mathcal{P}_0: \Pi_1(M) \to \mathcal{C} \)

the trivial representation

\[ \mathcal{P}_0 \text{ acyclic } \iff b_k(M) = 0 \quad \forall k \geq 0 \]

\( \Rightarrow \) The above result has implications on \( \pi_1(M) \) only if all Betti numbers vanish.
Thm. (Cekić–Dyatlov–Paternain–K. 2020): Let $(\Sigma, g)$ be a compact connected orientable hyperbolic 3-manifold. Then there is an open and dense set $U \subset C^\infty(\Sigma, \mathbb{R})$ such that for each $f \in U$ there is an $\varepsilon > 0$ such that the geodesic flow $\varphi^{g_t}_+ : S^1_{g_t}\Sigma \to S^1_{g_t}\Sigma$ of the metric $g_t := e^{tf}g$ satisfies

$$n(g_t) = n(\varphi^{g_t}_+) = \begin{cases} 4 - 2b_1(\Sigma), & \tau = 0 \\ 4 - b_1(\Sigma), & \tau \neq 0, \quad |\tau| \leq \varepsilon. \end{cases}$$

$\Rightarrow$ If $b_1(\Sigma) > 0$, then the function $\{\text{metrics on } \Sigma \text{ with } \tau < 0\} \ni g \mapsto n(g) \in \mathbb{Z}$ jumps at $g = g_{\text{hyperbolic}}$ along any generic conformal perturbation of $g_{\text{hyperbolic}}$. 

$\Rightarrow$ $n(\varphi_t)$ no longer a topological invariant, seems "sensitive to symmetries"
In particular, \( n(\gamma_+^t) \) can jump under continuous perturbations within the class of contact Anosov flows. More precisely:

**Thm. (Cekić-Dyatlov-Paternain-K. 2020):** Let \( M = S\Sigma \), where \( \Sigma \) is an oriented compact connected hyperbolic 3-manifold. Let \( U \subset M \) be a non-empty open set. Then there is an open, dense set \( \mathcal{U} \subset \mathcal{C}^0(U, \mathbb{R}) \subset \mathcal{C}^0(M, \mathbb{R}) \) such that for every \( \mathbf{f} \in \mathcal{U} \) \( \exists \varepsilon > 0 \) such that the Reeb flow \( \gamma_+^t \) of the contact form \( \alpha_\varepsilon := e^{\frac{\varepsilon}{t}} \mathbf{f} \) satisfies:

\[
\begin{align*}
    n(\alpha_\varepsilon) &= n(\gamma_+^t) = \\
    &= \begin{cases} 
        4 - 2b_1(\Sigma), & \varepsilon = 0, \\
        4 - b_1(\Sigma), & \varepsilon \neq 0, |\varepsilon| < \varepsilon. 
    \end{cases}
\end{align*}
\]

Easier to prove than the previous theorem.

First results exhibiting jumps of \( n(\gamma_+^t) \) within the classes of contact and geodesic Anosov flows.
3) Pollicott-Ruelle resonant states

M cpt., connected, $\gamma_t : M \to M$ Anosov flow with generator $X \in C^\infty(M; TM)$,

$T^*M = E^*_0 \oplus E^*_s \oplus E^*_u$, $E^*_s, E^*_u$ orientable

$T^*M = E^*_0 \oplus E^*_s \oplus E^*_u$, $E^*_0 (E^*_s \oplus E^*_u) = 0$, $E^*_s (E^*_0 \oplus E^*_u) = 0$, $E^*_u (E^*_0 \oplus E^*_u) = 0$

$\mathcal{S}^k := (\Lambda^k T^*M)_c, k \in \mathbb{N}_0$, $\mathcal{S}_0 := \{ w \in \mathcal{S}^k : \lambda_x w = 0 \} \cong (\Lambda^k (E^*_s \oplus E^*_u))_c$

$D^*_k (M; \mathcal{S}^k) := \{ v \in D^*_k (M; \mathcal{S}^k) : \text{WF}(v) \subset E^*_u \}$, similarly with $\mathcal{S}_0$, $E^*_u$

$\text{Res}^k_0 := \{ v \in D^*_k (M; \mathcal{S}^k) : \lambda_x v = 0 \}$

resonant states, $\dim_\mathbb{C} \text{Res}^k_0$ : geometric multiplicity

$\text{Res}^{k, \ell}_0 := \bigcup_{\ell \geq 0} \text{Res}^{k, \ell}_0$, $M_{k, 0} := \dim_\mathbb{C} \text{Res}^{k, \infty}_0$ : algebraic multiplicity

generalized resonant states
By a factorization argument and using microlocal machinery, one finds
\[ n(\Psi_+) = \sum_{k=0}^{\dim M-1} (-1)^k m_{k,0} \]  
(see Dyatlov-Zworski 2016)

If geometric and algebraic multiplicities coincide, say **semisimplicity** holds

For \( k = \dim M-1 \) and \( k=0 \), semisimplicity always holds and \( m_{0,0} = m_{\dim M-n,0} = 1 \)

Now, suppose \((M, \alpha)\) contact mfd., \( \dim M = 5 \), \( \Psi_+ \) contact Anosov.

Then \( \land \alpha : \text{Res}_0^2 \to \text{Res}_0^1 \) is an isomorphism

\[ \Rightarrow n(\Psi_+) = 2 - 2m_{1,0} + m_{2,0} \]

\( \Rightarrow \) need to understand \( m_{k,0} \) for \( k=1,2 \), requires understanding of semisimplicity
Important tool to study semisimplicity: resonant/co-resonant state pairings
\[ \langle \cdot, \cdot \rangle : E^u(M; J_0) \times E^s(M; J_0^{-1}) \to \mathbb{C}, \quad (u, v) \mapsto \sum_{M} \langle u, v \rangle \]
eextends to
\[ \langle \cdot, \cdot \rangle : D^\prime_{E^u_0}(M; J_0^k) \times D^\prime_{E^s_0}(M; J_0^{k^{-1}}) \to \mathbb{C}. \]

\[ \text{Res}^k_{\cdot, \cdot} := \{ v \in D^\prime_{E^s_0}(M; J_0^k) : x^\nu = 0 \}, \quad \text{similarly: Res}^k_{\cdot, \cdot}, \text{Res}^k_{\cdot, \cdot} \]co-resonant states

**Basic observation:** semisimplicity holds for \( K \) iff \( \langle \cdot, \cdot \rangle \) restricts to a non-degenerate pairing on \( \text{Res}^k_{\cdot, \cdot} \times \text{Res}^{k^{-1}}_{\cdot, \cdot} \)

**Involution** \( J : M = S \Sigma \to M \) induces isomorphism \( \text{Res}^k_{\cdot, \cdot} \cong \text{Res}^k_{\cdot, \cdot} \) for \( k, k^{-1} \).
Important role played by smooth resonant states / smooth representants:
\[ d\alpha \in \text{Res}_0^2 \cap C^\infty(M;\mathbb{R}^2) \]

If \( u \in \text{Res}_0^k \), \( du \in C^\infty(M;\mathbb{R}^{2k}) \), \( \exists \tilde{u} \in C^\infty(M;\mathbb{R}^k) \), \( \forall \epsilon > 0 \):

\[ u = \tilde{u} + dw \]

get map \( \Pi_k: \text{Res}_0^k \cap \ker d \to H^k(M;\mathbb{C}) \)

\[ u \mapsto [\tilde{u}] \]

Hamenstädt 1985: If \( M = \Sigma \), \( \Sigma \) cpt. orient. 3-dim Riem. mfd, with \( \kappa < 0 \),

\( \gamma \) geodesic flow, and \( \dim_c \text{Res}_0^2 \cap C^\infty(M;\mathbb{R}^2) > 1 \),

then \( \kappa = \text{const} \), i.e., \( \Sigma \) is hyperbolic up to constant rescaling.

On the other hand, if \( M = \Sigma \) s.t. \( \kappa > -1 \), \( \exists \psi \in \text{Res}_0^2 \cap C^\infty(M;\mathbb{R}^2) \) \((\text{Cdd})\)
Let now $M = S^2$ with $(\Sigma, g)$ hyperbolic, 3-dim., oriented, connected.

Then we prove:
- $\Psi$ is closed but not exact ($\Psi$ represents the Euler class of $T\Sigma$)
  - $\dim \text{Res}_0^2 \cap \text{Ker } d = b_7(M)$, $\dim \text{Res}_0^7 = 2b_7(M) = m_{7,0}$ (semisimplicity holds)
  - $\dim \text{Res}_0^2 = b_7(M) + 2$, $m_{2,0} = 2b_7(M) + 2$ (semisimplicity fails if $b_7(M) > 0$)

- the range of the map $L_x : \text{Res}_0^2 \to \text{Res}_0^5$ is equal to $d(\text{Res}_0^7)$

- the existence of $\Psi$ and of non-closed elements in $\text{Res}_0^7$ is due to the existence of an involution $I : TM \to TM$ annihilating $E_0$, rotating by $\frac{\pi}{2}$ in $E_5, E_6$, commuting with $d\Psi$:
  $$\Psi|_{(x,v)} (3,y) = d\omega|_{(x,v)} (I(x,v)3, y)$$

I corresponds to Hodge star on sphere $S^2$ = $\partial_0 \mathbb{H}^3$ at infinity of $\Sigma = \mathbb{H}^3$. 
Contact perturbations

\[ \alpha_{\varepsilon} \in \mathfrak{e}^{\omega}(M; TM), \quad \tau \in (-\varepsilon, \varepsilon), \quad \alpha_0 \text{ std. contact form on } M = \Sigma \cong S^* \Sigma \]

\[ X_{\varepsilon} \in \mathfrak{e}^{\omega}(M; TM) \] Reeb vector field of \( \alpha_{\varepsilon} \); \( \varphi_{\varepsilon}^t : M \to M \) flow of \( X_{\varepsilon} \)

For \( \varepsilon > 0 \) small enough, \( \alpha_{\varepsilon} \) is contact and \( X_{\varepsilon} \) is Anosov \( \forall \varepsilon \in (-\varepsilon, \varepsilon) \)

\[ \beta := \partial_x \alpha_{\varepsilon} \bigg|_{t=0} \in \mathfrak{e}^{\omega}(M; T^*M) \]

Then \( \langle L_{X_{\varepsilon}} \beta, \cdot \rangle : \mathfrak{d}(\text{Res}_0) \times \mathfrak{d}(\text{Res}_{0, \chi_0}) \to \mathbb{C} \) is non-degenerate.

\[ n(\varphi_{\varepsilon}^t) = 4 - b_1(\Sigma) \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\} \quad \text{for some } \varepsilon_0 > 0. \]

Proof methods: Study continuity of resolvents \( R_k^\varepsilon (\mathbb{H}) := (\varphi_{\varepsilon}^t - 2)^{-1} \); \( P_{\varepsilon}^k := -i \varphi_{\varepsilon}^t 2 \).

on appropriate \( \varepsilon \)-independent function spaces
$H_{Y_{6,s}}(M;\mathbb{R}^k) := \mathcal{E}^r_{0}(\mathcal{E}) H^{s}(M;\mathbb{R}^k) \quad \text{for } r \geq 0, \text{ s} \in \mathbb{R}$

$G(p, \beta) = m(p, \beta)(1 + |\beta|)$ as in Bonthonneau '20

$D^E_{Y_{6,s}}(M;\mathbb{R}^k) := \{v \in H_{Y_{6,s}}(M;\mathbb{R}^k) \mid P^K v \in H_{Y_{6,s}}(M;\mathbb{R}^k) \}$

For $\varepsilon_0 > 0$ small $\exists c_0$ s.t. for $\gamma > c_0 + 151$, $\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\text{Im} \varepsilon > -1$,

$P^K_{\varepsilon} : \quad D^E_{Y_{6,s}}(M;\mathbb{R}^k) \rightarrow H_{Y_{6,s}}(M;\mathbb{R}^k)$ is Fredholm

and the resolvent $R^E_{\varepsilon}(\gamma) : H_{Y_{6,s}}(M;\mathbb{R}^k) \rightarrow H_{Y_{6,s}}(M;\mathbb{R}^k)$ is bounded locally uniformly in $\gamma$, outside the set $\{ (\varepsilon, \gamma) \in (-\varepsilon_0, \varepsilon_0) \times \mathbb{C} : \gamma \text{ is a resonance of } P^K_{\varepsilon} \}$

which is closed in $(-\varepsilon_0, \varepsilon_0) \times \mathbb{C}$

(Cekić-Potemkin '19)
The proof of the main result for contact perturbations then reduces to showing:

\[ \forall u \in Res_0, v \in Res_\alpha, \ du \neq 0, \ dv \neq 0 : \]

\[ \text{supp}(\alpha \wedge du \wedge dv) = M. \]

(In unperturbed hyperbolic situation)

Using horocyclic invariance and pullbacks along transport to \( S^2 \) at infinity, the statement reduces to the following:

\[ \forall g_+, g_- \in D'(S^2), \ \text{supp}(g_+) = S^2; \]

\[ \text{supp} \ g_+ \otimes g_- = S^2 \times S^2. \]
For the main result on metric perturbations, need to prove:

Given $f \in D'_E(M) \otimes \mathbb{R}$ s.t. $xf + 2f = 0$, one has

$$S_a \Pi_* (ff^*) \, dv_{\Sigma} \neq 0 \quad \text{for generic } a \in E^0(\Sigma),$$

where $\Sigma$.

$\Pi_* : D'(M) \to D'(\Sigma)$ is induced by integration over the fiber.

$f^* = f^* \mathcal{J} \mathcal{J} : M \to M, \quad (x, \mathcal{J}) \mapsto (x, -\mathcal{J})$

Strategy:

- Assign harmonic 1-forms $u, u^* \in \Omega^1(\Sigma)$ to $f, f^*$
  s.t. $u = 0 \Rightarrow f = 0$

- Use a convolution operator to relate $u, u^*$ with $\Pi_* (ff^*)$
The operator \( Q_s : C_c^\infty (\mathbb{H}^3) \rightarrow C_c^\infty (\mathbb{H}^3) \),

\[
f \mapsto \int_{\mathbb{H}^3} K_s(x,z) f(z) \, d\text{vol}_{\mathbb{H}^3}(z) =: (Q_s f)(x),
\]

extends for \( s > \frac{1}{2} \) to a smoothing operator \( Q_s : D'(\mathbb{H}^3) \rightarrow C_c^\infty (\mathbb{H}^3) \).

Main technical result:

\[
Q_4 \Pi_* (ff^*) = \frac{n}{24} \Delta (|u|^2)
\]

\[
\Rightarrow \text{ If } \Pi_* (ff^*) = 0, \text{ then } |u| = \text{const } \Rightarrow u = 0 \Rightarrow f = 0
\]

\( u \) harmonic 1-form